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Note

A Numerical Study of the Cusp Catastrophe for Bénard Convection in Tilted Cavities

In the Bénard problem a fluid is confined between horizontal surfaces with the lower surface maintained at a higher temperature than the upper one. The consequences are well understood; when the Rayleigh number Ra is below a certain value then there is no flow and heat is transferred by conduction alone. At the critical value there is a bifurcation from the conducting solution which, because of the vertical symmetry, is in the form of a pitchfork [1].

The pitchfork bifurcation in one parameter, Ra in this case, is unstable in the sense that small perturbations to the boundary conditions or equations, which break the symmetry, change the qualitative picture. By introducing a second, symmetry-breaking parameter into the problem, the pitchfork is unfolded to a cusp catastrophe, which is structurally stable [2]. The original pitchfork is just a section through the cusp.

From these considerations, we deduce that, when the surfaces in the Bénard problem are tilted at θ degrees to the horizontal, then the pitchfork is unfolded to a cusp catastrophe in the parameters Ra and θ. This is shown in Fig. 1, from which it is clear that the section at $\theta = 0$ through the cusp is the usual Bénard bifurcation diagram. For non-zero θ, a typical section gives the one-sided bifurcation shown in Fig. 2. In this case the flow develops smoothly from zero Rayleigh number to the lower branch. There are however two further solutions above a critical value of Ra which cannot be reached smoothly. The lower of these is unstable, but the upper branch is stable and is in principle observable.

In this letter, we predict the critical Rayleigh number for the appearance of these anomalous solutions as a function of angle of tilt θ. For a fixed tilt of 1° we also compute the anomalous solution for various Rayleigh numbers above the critical value.

We consider the particular case of confined Bénard convection in a 2-dimensional cavity with rigid sidewalls and aspect ratio of 1. The vertical walls are insulated. The equations for natural convection are solved in the Boussinesq approximation throughout. Three distinct methods are used to map the solution surface represented in Fig. 1.

(i) For fixed values of Ra and θ, the set of equations for natural convection, which we denote by

$$g(x, \mathrm{Ra}, \theta) = 0, \tag{1}$$

are discretised in a finite-element approximation and linearised by the

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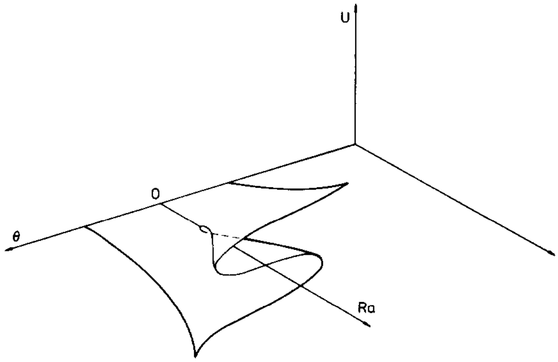

FIG 1. The cusp catastrophe for Bénard convection in a tilted cavity.

Newton–Raphson method [3]. In the above equation x denotes the solution vector, θ is the clockwise inclination of the heated surfaces to the horizontal, and the Rayleigh number Ra is based on the imposed temperature difference and the cavity height.

Euler–Newton continuation in either Ra or θ is then used to follow lines of constant θ or Ra on the surface of Fig. 1.

(ii) As we have already discussed, the vertical symmetry ensures that the cusp is located at $\theta = 0$ and is a pitchfork bifurcation with respect to the parameter Ra. This means that we can locate the cusp using an algorithm for finding bifurcation points due to Moore [4], which involves the solution of the equations:

$$\begin{aligned} g(x, \mathrm{Ra}, 0) + \Delta\psi &= 0, \\ \psi^T g_x(x, \mathrm{Ra}, 0) &= 0, \\ \psi^T g_{\mathrm{Ra}}(x, \mathrm{Ra}, 0) &= 0, \\ \psi^T \psi &= 1. \end{aligned} \tag{2}$$

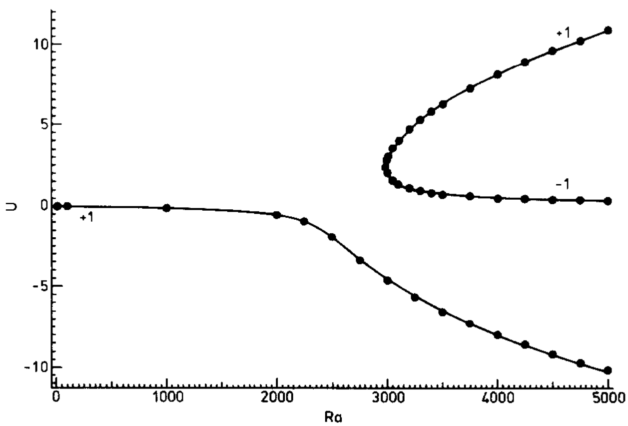

FIG. 2. State diagram for cavity tilted by 1°. The component of velocity parallel to the top of the cavity at (0.5, 0.9) is plotted against the Rayleigh number.

In these equations ψ is the left eigenvector of g_x, a subscript denotes differentiation with respect to that subscript, and Δ is a parameter introduced for numerical convenience, since otherwise the system is overdetermined. In the present case symmetry ensures that $\Delta = 0$ to machine accuracy. These equations were also discretised in the finite-element approximation and solved by Newton's method. The solution of these equations was used as the initial guess for the calculation of the limit point at the first non-zero value of tilt considered.

(iii) The limit (one-sided bifurcation) points were found from the solution of the extended set of equations proposed by Moore and Spence [5].

$$\begin{aligned} g(x, \mathrm{Ra}, \theta) &= 0, \\ g_x(x, \mathrm{Ra}, \theta)\,\phi &= 0, \\ l(\phi) &= 1, \end{aligned} \tag{3}$$

where ϕ is the right eigenvector of g_x and l is a linear functional which is introduced to normalise ϕ. For a given value of one parameter, θ say, we again discretise these equations in the finite-element approximation and solve them by Newton's method to give the value of the other parameter, Ra, at which there is a limit point. We then use Euler–Newton continuation in the first parameter to obtain the variation of the limit points with θ, thus tracing the folds in the solution surface represented in Fig. 1. This also gives the solution x at the limit point, so that it is straightforward to step onto either the upper or the lower anomalous solution by solving Eq. (1).

Figure 2 shows a state diagram generated by the above techniques, for a 1° angle of tilt. The circles are computed values and have been joined smoothly by a spline interpolation. The labels ± 1 are the predicted values of the Leray–Schauder index [6], and these are in agreement with the expected stability properties, a negative sign indicating unstable steady flow [6]. In the present case, the branches with positive index are stable. The anomalous solution on the upper branch is similar to the normal solution but with opposite sense of rotation. The anomalous streamlines and isotherms at a Rayleigh number of 5,000 are shown in Fig. 3.

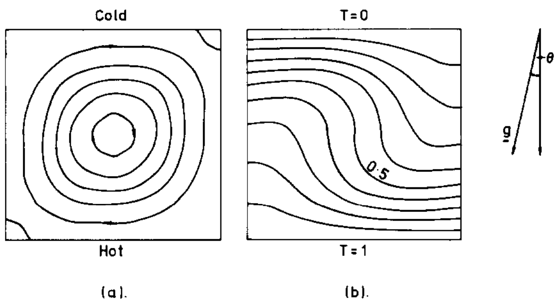

FIG. 3. Streamlines (a) and isotherms (b) for the anomalous solution at a Rayleigh number of 5,000 and 1° tilt.

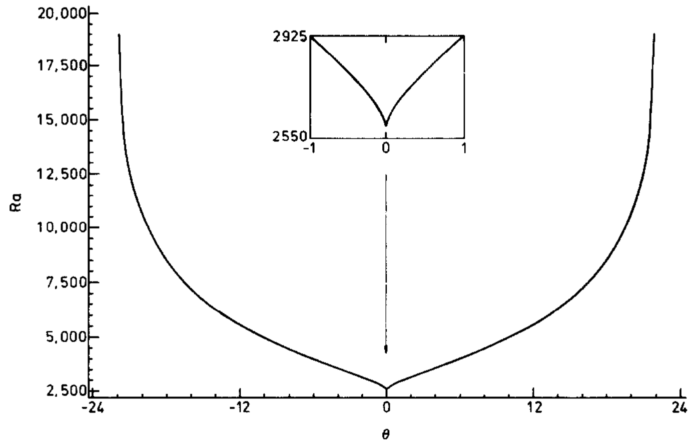

FIG. 4. Locus of limit points as a function of Ra and θ.

Figure 4 shows the locus of limit points as a function of Ra and θ. This is the projection of the folds of Fig. 1 onto the Ra–θ plane and shows the critical Rayleigh number for the appearance of the anomalous solution at any tilt. It is of course symmetric about $\theta = 0$, and the cusp is evident over a rather narrow range of angles, of $\pm 1°$. However, the striking feature of Fig. 4 is that the curve is asymptotic to the lines $\theta = \pm 22.0°$, so that the anomalous solutions exist only for angles of tilt less than this limiting value.

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